MATH 2028 Generalized Stokes Theorem

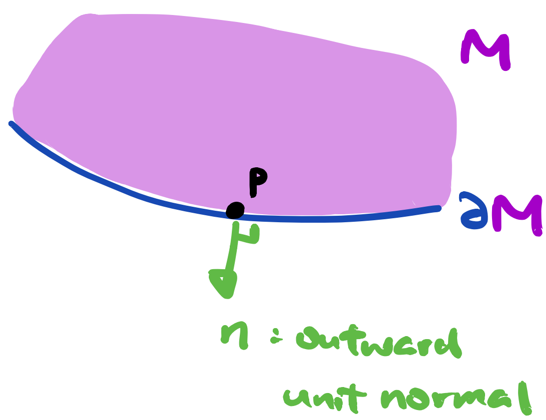
Goal: State and prove the general Stokes' Thm for submanifolds in $\mathbb{R}^n$.

Theorem (Stokes' Thm)

Let $M \subseteq \mathbb{R}^n$ be a compact, oriented, k-dim'l submanifold with boundary ∂M, equipped with the induced "boundary orientation". **THEN**:

$$(*) \qquad \boxed{\int_M d\omega = \int_{\partial M} \omega} \qquad \forall\ (k-1)\text{-forms } \omega \text{ in } \mathbb{R}^n$$

Boundary orientation:



A basis $\{v_1, \ldots, v_{k-1}\}$ for $T_p\partial M$ is positively oriented

$$\Updownarrow$$

The basis $\{n, v_1, \ldots, v_{k-1}\}$ for T_pM is positively oriented w.r.t. the orientation on M

Proof of Stokes' Theorem

Since (*) is linear in ω on both sides, by the definition using partition of unity, we can simply consider the case that $\mathrm{Spt}(\omega)$ is contained inside a parametrization $g : \mathcal{U} \to \mathbb{R}^n$ where $\mathcal{U}$ is an open subset of either $\mathbb{R}^k$ or $\mathbb{R}^k_+$.

By definition,

$$\int_M d\omega = \int_{\mathcal{U}} g^*(d\omega) = \int_{\mathcal{U}} d(g^*\omega)$$

Here, $g^*\omega$ is a $(k-1)$-form on $\mathcal{U} \subseteq \mathbb{R}^k$.

$$g^*\omega = \sum_{i=1}^{k} f_i(x)\, dx_1 \wedge \cdots \wedge \underbrace{\widehat{dx_i}}_{\text{this term taken away}} \wedge \cdots \wedge dx_k$$

Hence, the exterior derivative is

$$d(g^*\omega) = \left(\sum_{i=1}^{k} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_k$$

which is a k-form on $\mathcal{U} \subseteq \mathbb{R}^k$

Case 1 : $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^k$ (i.e. $\omega = 0$ on ∂M)

We want to show

$$\int_M d\omega = \int_{\mathcal{U}} d(g^*\omega) = 0$$

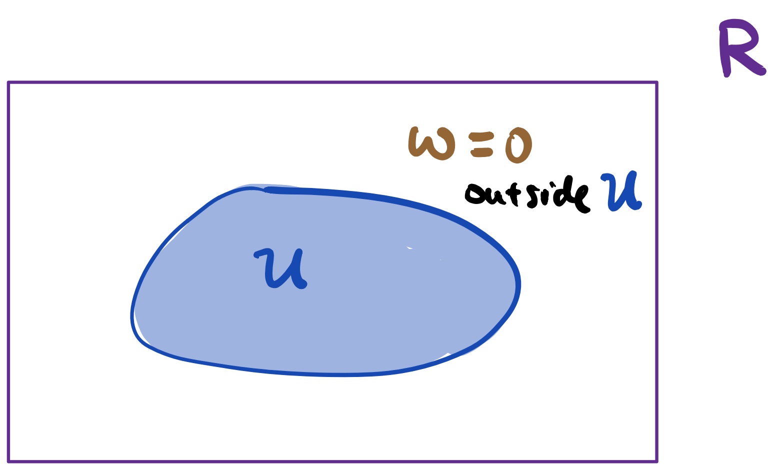
Recall that $\operatorname{spt}(g^*\omega) \subseteq \mathcal{U}$, we can extend it smoothly to be identically zero outside $\mathcal{U}$.

Take a rectangle $R = [a_1, b_1] \times \cdots \times [a_k, b_k] \supseteq \mathcal{U}$.

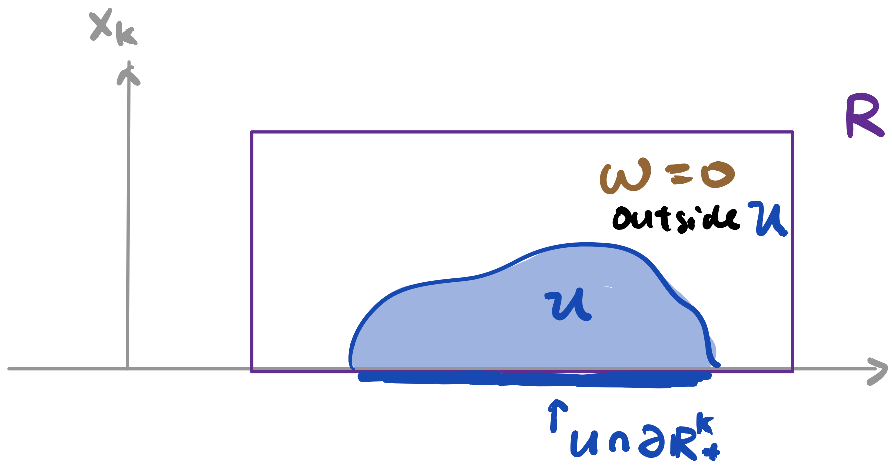
$$\int_M d(g^*\omega) = \int_R \left(\sum_{i=1}^{k} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_k$$

$$\overset{\text{Fubini}}{=} \sum_{i=1}^{k} (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \underbrace{\left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i \right)}_{=0} dx_1 \cdots \widehat{dx_i} \cdots dx_k$$

$$= 0$$



Case 2 : $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^k_+$ (i.e. we have the following picture)



The same calculation then yields

$$\int_M d(g^* \omega) = \int_R \left(\sum_{i=1}^{k} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_k$$

$$\overset{\text{Fubini?}}{=} \sum_{i=1}^{k} (-1)^{i-1} \int_0^{b_k} \cdots \int_{a_1}^{b_1} \underbrace{\left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i \right)}_{= 0 \text{ except for } i = k} dx_1 \cdots \widehat{dx_i} \cdots dx_k$$

$$= (-1)^{k-1} \int_{a_{k-1}}^{b_{k-1}} \cdots \int_{a_1}^{b_1} \left(\int_0^{b_k} \frac{\partial f_k}{\partial x_k} dx_k \right) dx_1 \cdots dx_{k-1}$$

$$= (-1)^k \int_{\mathcal{U} \cap \partial \mathbb{R}^k_+} f_k(x_1, \ldots, x_{k-1}, 0)\, dx_1 \cdots dx_{k-1}$$

boundary orientation

$$= \int_{\mathcal{U} \cap \partial \mathbb{R}^k_+} g^* \omega = \int_{\partial M} \omega$$

$\blacksquare$